THE DEGREE AND CLASS OF MULTIPLY TRANSITIVE GROUPS, II*

BY

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1. The paper to which this is a sequel had to do only with those multiply transitive groups of class $\mu(>3)$ in which at least one substitution of degree μ is of even order.† Among other results, it was there proved that the degree n of triply transitive groups of class $\mu(>3)$, which contain a substitution of even order on μ letters, does not exceed 2μ. Triply transitive groups of class μ and degree 2μ exist. This measure of success was due to the simplicity of structure of diedral rotation groups of class μ generated by two similar substitutions of degree μ and order 2. Up to the present time no better limit than that of Bochert, $n \leqq 3\mu(\mu>6)$, has been found to apply to these triply transitive groups of class $\mu(>3)$ in which all substitutions of degree μ are of odd order.‡ But it will here be proved that if such a group contains a substitution of order p^c (p an odd prime) on μ letters,

$$\mu > \frac{n}{2}\left(1 - \frac{1}{p^c}\right) - \frac{2}{p^c}\cdot$$

For doubly transitive groups the lower limit for the class $\mu(>3)$ in terms of the degree when there is a substitution of even order on μ letters present in the group is given by

$$\mu > \frac{n}{2} - \frac{n^{1/2}}{2} - 1.\text{§}$$

Whenever applicable, this replaces Bochert's limit of $n/3-2n^{1/2}/3$.¶ We can now prove that, when a doubly transitive group of class $\mu(>3)$ contains a substitution of odd prime order p, and (when $p=3$) n is sufficiently large,

$$\mu > \frac{n}{2}\left(1 - \frac{1}{p}\right) - \frac{n^{1/2}}{2}\left(1 - \frac{1}{p^2}\right)^{1/2} - 1.$$

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† Manning, these Transactions, vol. 18 (1917), p. 463.

‡ Bochert, Mathematische Annalen, vol. 40 (1892), p. 185.

§ Manning, Bulletin of the American Mathematical Society, vol. 20 (1914), p. 468.

¶ Bochert, Mathematische Annalen, vol. 49 (1897), p. 144.

For $p=3$, this is only $\mu>n/3-(2n)^{1/2}/3-1$. It was stated by Bochert without proof that $n/3-(2n)^{1/2}/3$ is a limit to which his ascending series of inferior limits for μ approaches on repeated use of his inequality (19). That is a mistake. His inequality is satisfied by $n/3-(2n)^{1/2}/3$ and also by $n/3-(2n)^{1/2}/3-1/2$ while a true inferior limit for μ is a number that does not satisfy the given inequality and which therefore is less than $n/3-(2n)^{1/2}-1/2$.

2. Bochert's Lemma will be used in the following form:

If the substitutions S and T have exactly m letters in common, and if S replaces q, and T r, common letters by common letters, the degree of $S^{-1}T^{-1}ST$ is not greater than $3m-q-r$.

This lemma remains indispensable in studying quadruply transitive groups of class $\mu(>3)$ and those doubly and triply transitive groups of class $\mu(>3)$ in which all the substitutions of degree μ are of order 3. But additional information is given in other cases by the following lemma:

LEMMA. *If S and T are two substitutions of degree μ and odd order d which generate a group of class μ, and if no power of S is commutative with a power of T (identity excepted), S and T have at least $\mu/2-\mu/(2d)$ letters in common.*

Let S and T have exactly m letters in common, m roman letters, say. Let the other letters of S and T be greek letters. From Bochert's Lemma it is known that $m\geqq[\mu/3]$, the integral part of $\mu/3$. Then no proof is needed when $d=3$. We assume that $d\geqq 5$. Now we say that S has s_i cycles each of which contains i roman letters, and that T has t_j cycles each of which contains j romans. Then

$$s_0+s_1+\cdots+s_d=t_0+t_1+\cdots+t_d=\frac{\mu}{d},$$

the number of cycles in S and T. Also

$$\begin{aligned} s_1+2s_2+3s_3+\cdots+ds_d &= t_1+2t_2+3t_3+\cdots+dt_d \\ &= \mu/3-\epsilon+k_1=\mu/3+k, \end{aligned}$$

where $\epsilon=0$, $1/3$, or $2/3$, as required to make $\mu/3-\epsilon=[\mu/3]$, and k_1 is a positive integer or zero.

Consider a cycle of S in which there are i romans. It generates a regular cyclic group. Therefore in the $d-1$ powers of this cycle every sequence of two letters occurs once and only once; then in these $d-1$ powers of one cycle of S there are $i(i-1)$ roman sequences. In the $d-1$ powers of S the number of roman sequences is exactly $2s_2+6s_3+\cdots+i(i-1)s_i+\cdots+d(d-1)s_d$, and the average number of roman sequences in the $d-1$ powers of S is $[2s_2+6s_3+\cdots+d(d-1)s_d]/(d-1)$. Similarly, in the powers of T the average number of roman sequences is $[2t_2+6t_3+\cdots+d(d-1)t_d]/(d-1)$. From the powers of S and T let two substitutions S^u and T^v be chosen, each of which has the average or more than the average number of roman sequences. Since the number of roman letters is $\mu/3+k$, the total number of roman sequences in S^u and T^v jointly cannot exceed $3k$. For Bochert's Lemma asserts that the degree of $S^{-u}T^{-v}S^uT^v$ is at most $\mu+3k$ diminished by the number of roman sequences in S^u and T^v. And since S^u and T^v are not commutative this degree is at least μ. Then

$$\sum_{i=0}^{d}\frac{i(i-1)s_i}{d-1}+\sum_{j=0}^{d}\frac{j(j-1)t_j}{d-1}\leqq 3k,$$

and one of the two summation terms of this inequality is not greater than $3k/2$, say the first. Thus we have simultaneously

$$s_0+s_1+s_2+\cdots+s_d=\frac{\mu}{d},$$

$$s_1+2s_2+3s_3+\cdots+ds_d=\frac{\mu}{3}+k,$$

$$2s_2+6s_3+12s_4+\cdots+d(d-1)s_d\leqq\frac{3}{2}(d-1)k.$$

Let two terms, s_x and s_y, be eliminated $(x<y)$. The result is

$$\frac{(x+y-1)d-3xy}{3d}\mu+\sum_{i=0}^{d}(i-x)(i-y)s_i\leqq\frac{1}{2}(3d-2x-2y-1)k.$$

If now $x+1$ be put for y,

$$\sum_{i=0}^{d}(i-x)(i-x-1)s_i\geqq 0,$$

and therefore

$$\mu\leqq\frac{3d(3d-4x-3)}{2x(2d-3x-3)}k.$$

We may choose the integer x to make the coefficient of k a minimum. The continuous curve

$$y=\frac{3d(3d-4x-3)}{2x(2d-3x-3)}$$

has a minimum between $x=0$ and $x=2d/3-1$. From $D_xy=0$ we get $x=$

$\{3d-3-[(d+1)\ (d+3)]^{1/2}\}/4$, lying between $(d-3)/2$ and $(d-1)/2$, and to these two integers correspond equal ordinates, $6d/(d-3)$. Then

$$k \geqq \frac{d-3}{6d}\mu,$$

and

$$m \geqq \frac{d-1}{2d}\mu.$$

It follows that the degree of $\{S, T\}$ is at most $3\mu/2+\mu/(2d)$.

3. We prove the following theorem:

THEOREM I. *The class $\mu(>3)$ of a triply transitive group of degree n that contains a substitution of degree μ and of order p^c (p an odd prime) is greater than*

$$\frac{n}{2}\left(1-\frac{1}{p^c}\right)-\frac{2}{p^c}.$$

One of the substitutions of degree μ in the group G is $S=(a \cdots b \cdots) \cdots$, of order p^c. In case the exponent c is greater than unity, b is one of the letters in the same cycle of $S^{p^{c-1}}$ as a, and therefore in the same cycle as a in every power of S. There are in G subsitutions similar to S which displace a and fix b. Let $S_1, S_2, \cdots, S_w$ be a complete set of w such substitutions, conjugate under the transitive subgroup $G(a)(b)$ of G that fixes a and b. No power of S_i $(i=1, 2, \cdots, w)$ is commutative with a power of S. The substitutions of this complete set of conjugates displace exactly

$$w+\frac{w(\mu-1)(\mu-2)}{n-2}$$

letters of S.* For they all displace a, and each of the other $w(\mu-1)$ letters in the set $S_1, S_2, \cdots, S_w$ occurs as often as any other, that is, $w(\mu-1)/(n-2)$ times. Thus the $\mu-2$ letters of S, disregarding a and b, occur in the set $S_1, S_2, \cdots$ the number of times stated above. Then, by our lemma,

$$w+\frac{w(\mu-1)(\mu-2)}{n-2} \geqq w\left(\frac{\mu}{2}-\frac{\mu}{2p^c}\right),$$

whence

$$\frac{n-\mu}{p^c}-(\mu-2)\left(\frac{n}{2}-\frac{n}{2p^c}-\mu\right) \geqq 0.$$

* For a detailed explanation of this and of similar formulas which follow, see Bochert, Mathematische Annalen, vol. 40 (1892), p. 176.

If now we put $n/2-n/(2p^c)-\delta$ for μ in the left member of this inequality, we get

$$-\frac{n}{2}\left(\delta-\frac{\delta}{p^c}-\frac{1}{p^c}-\frac{1}{2p^{2c}}\right)+\delta^2+2\delta-\frac{\delta}{p^c}.$$

This is positive if $\delta=1/p^c$, but if $\delta=2/p^c$ it reduces to

$$-\frac{n}{2}\left(\frac{1}{p^c}-\frac{3}{p^{2c}}\right)+\frac{4}{p^c}+\frac{2}{p^{2c}},$$

negative if n is greater than $8+28/(p^c-3)$. It is known that there is no triply transitive group of class 5. Hence the theorem is true as stated without exception.

4. We prove the following theorem:

THEOREM II. *The class* $\mu(>3)$ *of a triply transitive group of degree* n *in which every substitution of degree* μ *is of order* 3 *is not less than* $(n+4)/3$.

Let $S=(abc)\cdots$, a substitution of degree μ of G. Let $S_1=(a\cdots)\cdots(b)$, similar to S. Under $G(a)(b)$, S_1 is one of w conjugates, no one of which is commutative with S. If S and S_i have exactly $\mu/3$ common letters (their roman letters), S_i can have no roman sequence. This fact is an immediate consequence of Bochert's Lemma. In other words, if S_i has a roman sequence, it has at least $\mu/3+1$ letters in common with S. If S and S_i have $\mu/3+1$ letters in common, S, because it has only $\mu/3$ cycles, has some two letters of S_i in sequence. Hence, if S_i has a cycle of three roman letters, which means three roman sequences, S and S_i have $\mu/3+2$ or more common letters. Now in $w/(n-2)$ substitutions of the set $S_1, S_2, \cdots, S_w$ the letter a is followed by a given letter of $G(a)(b)$. Then in $w(\mu-2)/(n-2)$ substitutions S_i, a is followed by a letter of S, and similarly a is preceded by a letter of S in $w(\mu-2)/(n-2)$ substitutions S_i. Therefore

$$w+\frac{w(\mu-1)(\mu-2)}{n-2}\geqq\frac{w\mu}{3}+2\,w\frac{\mu-2}{n-2}\cdot$$

This inequality reduces to

$$\frac{(\mu-2)(\mu-3)}{n-2}\geqq\frac{\mu-3}{3},$$

whence, because by hypothesis $\mu>3$,

$$n\leqq 3\mu-4.$$

5. We prove the following theorem:

THEOREM III. *The degree of a quadruply transitive group of class* μ (>3) *does not exceed* $2\mu+1$.

For quadruply transitive groups of class μ Bochert's limit is $2\mu+2$, while the limit found in the preceding paper under the restriction that one of the substitutions of degree μ is of even order is $2\mu-1$. In the following proof it is assumed that all the substitutions of degree μ in G are of odd order.

Two cases are to be distinguished: (1) At least one substitution of degree μ is of order >3. (2) All substitutions of degree μ are of order 3.

Case 1. Let $S=(a \cdots b \cdots)\cdots$ be a substitution of degree μ and of order >3. The letters a and b are in the same cycle of S but are not adjacent. There are w substitutions $S_1=(b)\ (a\cdots)\cdots$, $S_2, \cdots$, conjugate under $G(a)(b)$, a doubly transitive subgroup of G. It was shown in §3 that the number of times the letters of S are to be found in the set $S_1, S_2, \cdots$ is $w+w(\mu-1)(\mu-2)/(n-2)$. The number of times letters of S occur just before or just after a in the set $S_1, S_2, \cdots$ is $2w(\mu-2)/(n-2)$. The number of sequences in $S_1, S_2, \cdots$ is $w(\mu-2)$, if we exclude those of which one letter is a. Since $G(a)(b)$ is doubly transitive each sequence occurs as often as any other and therefore exactly $w(\mu-2)/[(n-2)(n-3)]$ times. Now there are $(\mu-2)(\mu-3)$ permutations two at a time of the letters of S (excluding a and b). Therefore $2w(\mu-2)/(n-2)+w(\mu-2)^2(\mu-3)/[(n-2)(n-3)]$ is the number of sequences in $S_1, S_2, \cdots$ of which both letters are letters of S. The number of times the letter before a (and the letter after a) in S occurs in the set $S_1, S_2, \cdots$ is $2w(\mu-1)/(n-2)$. Excluding a and b, S has $\mu-4$ sequences. Excluding a, S_i has $(\mu-1)(\mu-2)$ permutations of letters two at a time, and therefore the set $S_1, S_2, \cdots$ contains $w(\mu-1)(\mu-2)$ such permutations of $\mu-1$ letters two at a time, each occurring $w(\mu-1)(\mu-2)\cdot[(n-2)(n-3)]^{-1}$ times, so that each sequence of S (not including a or b) is in $w(\mu-1)(\mu-2)/[(n-2)(n-3)]$ subsitutions of the set $S_1, S_2, \cdots$. Therefore $2w(\mu-1)/(n-2)+w(\mu-1)(\mu-2)(\mu-4)/[(n-2)(n-3)]$ is the number of times permutations two at a time of letters of $S_1, S_2, \cdots$ are a sequence in S. Therefore the sum of the degrees of the w commutators $S^{-1}S_i^{-1}SS_i$ $(i=1, 2, \cdots, w)$ gives

$$3w+\frac{3w(\mu-1)(\mu-2)}{n-2}-\frac{2w(\mu-2)}{n-2}-\frac{w(\mu-2)^2(\mu-3)}{(n-2)(n-3)}$$
$$-\frac{2w(\mu-1)}{n-2}-\frac{w(\mu-1)(\mu-2)(\mu-4)}{(n-2)(n-3)}\geqq w\mu.$$

Whence, if z is put for $n-2\mu-3$,

$$(\mu - 3)[z^2 + (\mu + 5)z] + 2(\mu^2 - 10) \leqq 0,$$

or

$$z^2 + (\mu + 5)z + 2\mu + 6 \leqq 0 \quad (\mu > 5).$$

Therefore

$$n - 2\mu - 3 \leqq -2, \quad n \leqq 2\mu + 1.$$

Case 2. The substitutions of degree μ are all of order 3. We have $S = (abc) \cdots$ and the set of w conjugates under $G(a)(b)$: $S_1 = (b)(a \cdots) \cdots$, $S_2, \cdots$. Since $G(a)(b)$ is doubly transitive we can choose the two letters in one of these substitutions, in S_1 say, at will from the $n-2$ letters of $G(a)(b)$. If $S_1 = (ac\alpha)(b) \cdots$, $S^{-1}S_1^{-1}SS_1 = (c\alpha)(ab) \cdots$. If this substitution is of degree μ, $n \leqq 2\mu - 1$, and the theorem is true. Hence the degree of this commutator is not less than $\mu+1$. Also if $S_1 = (a\alpha c)(b) \cdots$, $S^{-1}S_1^{-1}SS_1 = (ac)(b\alpha) \cdots$, which must be of degree $\geqq \mu+1$.

In the first cycle of each of the substitutions $S_1, S_2, \cdots$ there occurs every possible sequence of the letters of $G(a)(b)$, and any such sequence occurs $w/[(n-2)(n-3)]$ times. Then the $2(n-\mu)$ sequences in which the letter c of S is followed or preceded by one of the $n-\mu$ letters of G fixed by S occur $2w(n-\mu)/[(n-2)(n-3)]$ times in the first cycle of the substitutions S_1, $S_2, \cdots$. Thus the commutator of S and $2w(n-\mu)/[(n-2)(n-3)]$ substitutions of the set $S_1, S_2, \cdots$ is of degree $\geqq \mu+1$.

In the set $S_1, S_2, \cdots, a$ is a letter of $2w(\mu-2)/(n-2)$ sequences in which the other letter is in S, and c occurs in $w(\mu-1)/(n-2)$ of the substitutions $S_1, S_2, \cdots$. In the set $S_1, S_2, \cdots$ there are $w(\mu-2)^2(\mu-3)/[(n-2)(n-3)]$ sequences, not involving a, whose two letters are the $\mu-2$ letters $c, d, \cdots$ of S. There are $\mu-3$ sequences in S (not including a or b) and each such permutation of letters two at a time occurs in $S_1, S_2, \cdots, w(\mu-1)(\mu-2) \cdot [(n-2)(n-3)]^{-1}$ times. Hence

$$3w + \frac{3w(\mu-1)(\mu-2)}{n-2} - \frac{2w(\mu-2)}{n-2} - \frac{w(\mu-1)}{n-2} - \frac{w(\mu-2)^2(\mu-3)}{(n-2)(n-3)}$$
$$- \frac{w(\mu-1)(\mu-2)(\mu-3)}{(n-2)(n-3)} \geqq w\mu + \frac{2w(n-\mu)}{(n-2)(n-3)},$$

or, with $z = n - 2\mu - 3$,

$$z^2 + (\mu + 4)z + \mu + 4 \leqq 0,$$

and therefore

$$n \leqq 2\mu + 1.$$

6. We prove the following theorem:

THEOREM IV. *Let μ be the class and n the degree of a doubly transitive group in which one of the substitutions of degree $\mu(>3)$ is of prime order $p(>3)$; then*

$$\mu > \frac{n}{2}\left(1 - \frac{1}{p}\right) - \frac{n^{1/2}}{2}\left(1 - \frac{1}{p^2}\right)^{1/2} - 1.$$

There is a substitution $S=(ab\cdots)\cdots$ of prime order p (>3) in the given doubly transitive group G, and S is one of w conjugates under G. Any sequence of two letters, as ab, occurs $w\mu/[n(n-1)]$ times in this set. The number of possible sequences in n letters, in which one letter belongs to S and the other does not, is $2\mu(n-\mu)$. Then in the complete set of w conjugate substitutions such sequences as $a\alpha$ and αa occur in all $2w\mu^2(n-\mu)\cdot[n(n-1)]^{-1}$ times. There can be at most $p-1$ such sequences in a cycle. Let us say that $S_1, S_2, \cdots, S_y$ are y substitutions of the set not commutative with S. Then

$$y \geqq \frac{2w\mu(n-\mu)}{n(n-1)(1-1/p)}.$$

We recall that if one of the w conjugate substitutions has exactly x letters in common with S,

$$\sum_w x = \frac{w\mu^2}{n},$$

$$\sum_w x(x-1) = \frac{w\mu^2(\mu-1)^2}{n(n-1)},$$

and

$$\sum_w \left(x - \frac{\mu^2}{n}\right)^2 = \frac{w\mu^2(n-\mu)^2}{n^2(n-1)}.$$

Now by our lemma, if S and S_i $(i=1, 2, \cdots, y)$ have x_i letters in common,

$$x_i \geqq \frac{\mu}{2}\left(1 - \frac{1}{p}\right),$$

and therefore

$$\sum_{i=1}^{y} x_i \geqq \frac{y\mu}{2}\left(1 - \frac{1}{p}\right).$$

Let us now restrict our attention to those doubly transitive groups for which

$$\mu \leqq \frac{n}{2}\left(1 - \frac{1}{p}\right).$$

For such groups

$$\frac{\mu}{2}\left(1 - \frac{1}{p}\right) - \frac{\mu}{2}\,\frac{2\mu}{n} \geqq 0,$$

and therefore

$$\sum_{i=1}^{y}\left(x_i - \frac{\mu^2}{n}\right)^2 \geqq y\left(\frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n}\right)^2.$$

Next,

$$\sum_{w-y}\left(x - \frac{\mu^2}{n}\right)^2 \leqq \frac{w\mu^2(n-\mu)^2}{n^2(n-1)} - y\left(\frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n}\right)^2.$$

Also

$$\sum_{w-y}\left(x - \frac{\mu^2}{n}\right)^2 \geqq \frac{1}{w-y}\left[\sum_{w-y}\left(x - \frac{\mu^2}{n}\right)\right]^2$$

$$\geqq \frac{1}{w-y}\left(\frac{y\mu}{2} - \frac{y\mu}{2p} - \frac{y\mu^2}{n}\right)^2,$$

so that

$$\frac{w\mu^2(n-\mu)^2}{n^2(n-1)} - y\left(\frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n}\right)^2 \geqq \frac{y^2}{w-y}\left(\frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n}\right)^2,$$

or

$$\frac{w\mu^2(n-\mu)^2}{n^2(n-1)} - \frac{wy}{w-y}\left(\frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n}\right)^2 \geqq 0,$$

or

$$\left(\frac{w}{y} - 1\right)\frac{\mu^2(n-\mu)^2}{n^2(n-1)} - \left(\frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n}\right)^2 \geqq 0.$$

Substituting for w/y,

$$\frac{\mu(n-\mu)(1-1/p)}{2n} - \frac{\mu^2(n-\mu)^2}{n^2(n-1)} - \frac{\mu^2}{n^2}\left(\frac{n}{2} - \frac{n}{2p} - \mu\right)^2 \geqq 0,$$

or finally,

$$\frac{n(n-\mu)(1-1/p)}{2\mu} - \frac{(n-\mu)^2}{n-1} - \left(\frac{n}{2} - \frac{n}{2p} - \mu\right)^2 \geqq 0.$$

Let $n/2 - n/(2p) - \epsilon n^{1/2}/2 - \delta$ be substituted for μ in the final inequality. In the resulting polynomial in $n^{1/2}$ it will be seen that the coefficient of n^3 vanishes if $\epsilon = (1 - 1/p^2)^{1/2}$. With this value of ϵ agreed upon, the polynomial may be written

$$\left(2 + \frac{2}{p^2} - 4\delta + \frac{4\delta}{p}\right)\epsilon n^{5/2} + \left(\frac{2\epsilon^2}{p} + 8\delta - 4\delta^2 + \frac{4\delta^2}{p}\right)n^2$$

$$+ \left(8\delta + \frac{8\delta}{p} + 12\delta^2 - 2 + \frac{2}{p}\right)\epsilon n^{3/2} + \left(8\delta^2 + 8\delta + \frac{8\delta}{p} - 4 + \frac{4}{p}\right)\delta n.$$

If $\delta=\frac{1}{2}$, this polynomial is positive; if $\delta=1$, it reduces to

$$-\left(1-\frac{2}{p}-\frac{1}{p^2}\right)\epsilon n^{3/2}+\left(2+\frac{3}{p}-\frac{1}{p^3}\right)n+\left(9+\frac{5}{p}\right)\epsilon n^{1/2}+6+\frac{6}{p},$$

which is negative for $n>57$. Any possible cases of exception are covered by the known theorems on non-alternating primitive groups which contain substitutions of order p and degree qp ($q=1, 2, 3, 4$).*

7. The above proof is valid for $p=3$ until we come to the discussion of the polynomial

$$-\ n^{3/2}+10\cdot 2^{1/2}n+48n^{1/2}+27\cdot 2^{1/2},$$

which is negative when $n>292$. If however we put $\delta=4/3$, the polynomial is

$$-\ 3n^{3/2}+11\cdot 2^{1/2}n+77n^{1/2}+58\cdot 2^{1/2},$$

and this is negative if $n>73$. If $n=73$, 70, 69, 60, 59, $[n-(2n)^{1/2}-4]/3=$ 18.9, 18.1, 17.8, 15.0, 14.7, respectively. It can be shown that this limit holds for $n\leqq 73$. It is known to be true for doubly transitive groups of class 6, 9, and 12†. For groups of class 15 or 18, we can use the following theorem:

A primitive group that contains a substitution of order p and degree pq (p an odd prime, $p<q<2p+3$), contains a transitive subgroup the degree of which is not greater than the larger of the two numbers $pq+q^2-q$ and $2q^2-p^2$.‡

Thus our doubly transitive group of class 18 contains a transitive subgroup H of degree $\leqq$ 63, and we know from the proof of the theorem cited that the latter is generated by substitutions of order 3 and degree 18. We are concerned only with $n=70$, 71, 72, 73. Then G is more than doubly transitive unless the transitive subgroup H is imprimitive. Since H is generated by similar substitutions of order 3 and degree 18, its systems of imprimitivity are of two, three, or six letters. If H has systems of imprimitivity of six letters each, its degree is not greater than 60, and G has a doubly transitive subgroup H' of degree $\leqq 67$; and if the systems are of two or three letters only, the same is true.§ Thus G is more than doubly transitive and we should have (by §4) $\mu>n/3$.

If G is of class 15, H is of degree $\leqq 41$, and the doubly transitive subgroup H' is of degree $\leqq$ 46.

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* Manning, these Transactions, vol. 10 (1909), p. 247.

† Manning, these Transactions, vol. 6 (1905), p. 45; American Journal of Mathematics, vol. 35 (1913), p. 229.

‡ Manning, these Transactions, vol. 12 (1911), p. 382, §12.

§ Manning, these Transactions, vol. 7 (1906), p. 499; *Primitive Groups*, part 1, 1921, p. 93.

Our result for $p=3$ leaves much to be desired but is at any rate of the same form as Theorem IV:

THEOREM V. *Let* μ *be the class and* n (>292) *the degree of a doubly transitive group in which one of the substitutions of degree* $\mu(>3)$ *is of order* 3; *then*

$$\mu > \frac{n}{3} - \frac{(2n)^{1/2}}{3} - 1.$$

If $n \leqq 292$,

$$\mu > \frac{n}{3} - \frac{(2n)^{1/2}}{3} - \frac{4}{3}.$$

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